Scaling properties of a rice-pile model: Inertia and friction effects

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We present a rice-pile cellular automaton model that includes inertial and friction effects. This model is studied in one dimension, where the updating of metastable sites is done according to a stochastic dynamics governed by a probabilistic toppling parameter p that depends on the accumulated energy of moving grains. We investigate the scaling properties of the model using finite-size scaling analysis. The avalanche size, the lifetime, and the residence time distributions exhibit a power-law behavior. Their corresponding critical exponents, respectively, τ, y, and y_r, are not universal. They present continuous variation versus the parameters of the system. The maximal value of the critical exponent τ that our model gives is very close to the experimental one, $\tau=2.02$ [Frette *et al.*, Nature (London) **379**, 49 (1996)], and the probability distribution of the residence time is in good agreement with the experimental results. We note that the critical behavior is observed only in a certain range of parameter values of the system which correspond to low inertia and high friction.

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I. INTRODUCTION

Since its introduction by Bak, Tang, and Wiesenfeld, the concept of self-organized criticality (SOC) [1,2] has made considerable impact on a number of fields in natural and social sciences. The idea of SOC is generally illustrated conceptually with avalanches in a pile of sand grains. Many different kinds of sandpile models have been studied. These include models with discrete or continuous variables [3] and deterministic or stochastic toppling [4,5]. SOC is described for some extended dynamical systems which can evolve into a statistically stationary state where events, i.e., avalanches of all sizes, are observed with long-range spatial and temporal correlations between them. The stationary critical state can be reached after some transient time by adding grains one by one.

Comparison with real systems has proved to be a sound test for the theory and models. However, early experimental studies of real sandpiles led to clear disagreement with the numerical models: instead of the power-law behavior, bounded distributions of avalanche sizes were observed [6–9]. Against this background, recent one dimensional rice-pile experiments [10] have shown that under some conditions a real rice pile can self-organize into a critical state: A system of grains with large aspect ratio presents a self-organized critical state. The main cause of the emergence of the critical behavior was explained by the presence of grain anisotropy, which gives rise to a variety of packing configurations that develop large slopes. The anisotropy restricts the way the grains moved down the slope and can produce an increasing frictional contact between grains that is able to cancel the inertial effects. The results provided from the rice-pile experiments sparked a renewed interest in the study of sandpiles.

Several models of granular material, which are called rice-pile models [4,5], have been proposed in order to model the friction effects. The rice piles have been studied experimentally and theoretically using both analytical methods and numerical simulations [11–13]. The distribution of times the added grains spend inside the system has been investigated analytically [11,12]. On the other hand, many variations and extensions of the basic idea of the rice-pile model have been considered [5,14,16]. The friction and inertial effects were introduced implicitly in a stochastic toppling parameter or in dynamical critical slopes.

Indeed, the friction effects in rice piles have been studied through the introduction of randomness in the relaxation rule instead of in the deposition rule. The randomness was illustrated by introduction of a probability p of toppling [4,5,14–16]. The introduction of such a toppling probability is motivated by the dynamics of granular material. Due to the different shapes, stickiness, momentum effects, stacking details, etc. of the sand grains, it is possible that a given slope is stable at one time whereas the same slope is unstable at another time. To mimic all these complicated degrees of freedom stochastic toppling conditions have been implemented. Physically, the parameter p describes the friction between the rice grains, which comes directly from the observation that there exists a large range of slopes in the system, with the possibility of obtaining metastable packing configurations. The introduced parameter p represents the fact that there is a finite probability that the system reaches a new steady state where no toppling occurs [15].

The rice-pile model assumes a discrete space, $i=1,2\ldots,L$, from left to right, as well as a discrete field (the height of the pile, or the number of grains), and the grains are slowly added at a fixed position in a quasi-one-dimensional system. The grains pile up until the local slope somewhere is larger than a critical slope z_c; then the top grain may become unstable and be transferred to the next right column. In addition, the effects of gravity were modeled by the introduction of a second critical slope z_g. In terms of the height $h(i)$ and the local slope, defined as $z(i)=h(i)-h(i+1)$, these prescriptions are expressed by the following rules:

$$z(i) \leq z_c, \quad \text{no toppling},$$

$$z_c < z(i) \leq z_g, \quad \text{toppling with probability } p,$$

$$z(i) > z_g, \quad \text{toppling}.$$

It was found that the system belongs to the local linear interface universality class where it displays a critical behavior with a power-law distribution of avalanche sizes characterized by a critical exponent $\tau = 1.55 \pm 0.02$, signaling the existence of no characteristic scales for the avalanche process. Moreover, the concordance with experiments for avalanche properties is only qualitative. The key to modeling the different ingredients, which make the model critical in one dimension, and taking into account heterogeneities of real systems, is the implementation of the stochastic toppling parameter which incorporates explicitly or implicitly all dynamical degrees of freedom. As in real sandpile experiments, the randomness is internal in the rice-pile model, and the stochastic parameter p may be a complicated function of local slope and the packing of particles. However, by using an exponential form of the toppling probability p[16] that approaches 1 in the limit of large $z(i)$ in a smooth way, it was shown that the nonlocal unlimited [17] sandpile model presents nonuniversal behavior with a continuous variation of the critical exponent τ related to the distribution of the avalanche sizes.

Frette *et al.* [10] found that for less elongated grains the system did not evolve into a critical state. The reason for this result can be understood only if the role of the inertia in the dynamics is remembered. To the best of our knowledge, there have been only a few attempts to construct an inertial sandpile model. Prado and Olami [18] tried to simulate the experiment of Ref. [7] and chose to associate moving particles with a decrease in the local stability, assuming that the critical slope z_c is a decreasing function of the energy or the momentum, accumulated by a grain of sand during a sequence of topples. As a result, they found SOC behavior only for small system sizes. For large system sizes the size distribution maintains the power-law behavior for low values of inertia while for high values the size distribution is dominated by large avalanches, and the SOC behavior is lost. The preferred scale for large avalanches, for large grain inertia, was also observed in a coupled-map lattice model of rotated sandpiles [19].

On the other hand, the effect of inertia and dissipation on a sandpile subject to constant tilt in a rotating cylinder was studied using a local approach to sandpile dynamics [20]. The established coupled stochastic nonlinear dynamical equations contain a transfer term, which represents the inertia in an implicit way since it is a mechanism for amplifying sandpile avalanches. Within this approach the critical exponents were found to be nonuniversal. They depend on the kind of noise and the symmetries of the models.

In this paper, we develop a model to test the effects of microscopic details on the large-scale behavior. Inspired by the configurational disorder observed in piles of rice, we suggest a model to study the effects of inertia and friction and we present a way to incorporate these quantities in the rice-pile model in order to develop a more appropriate model as well as to reach the rice-pile experiment exponent given by $\tau = 2.02$ and reproduce the results obtained experimentally. Thus, we propose an exponential form of the toppling probability where the energy of the toppling particles benefits from the contribution of two terms: the kinetic and the potential energy. The randomness effect introduced in the system is then studied using Monte Carlo simulations. In Sec. II we define and give a brief description of our model while in Sec. III we present and discuss the results obtained by using Monte Carlo simulations and compare them with those established for other rice-pile models. The conclusion is given in Sec. IV.

II. THE MODEL

A one-dimensional sandpile is defined by a set of integer heights h_i, $i = 1, 2, \ldots, L$, or equivalently by the local slopes $z_i = h_i - h_{i+1}$. We simulate a system with open right-hand boundary, $z_L = h_L$, whereas the left-hand boundary is taken to be closed, $z_0 = 0$. Particles are deposited one grain at a time at the site $i = 1$, increasing its height by unity,

$$h(1) \to h(1) + 1. \tag{1}$$

The rate of deposition is slow enough that any avalanche initiated by an added grain will have ended before a new grain is deposited. In the local limited sandpile model [17], a site i becomes active when z_i becomes greater than the critical slope parameter z_c. A site i is considered active if, in the anterior time step, (1) it received a grain from column $(i-1)$, (2) it toppled a grain to site $(i+1)$, or (3) site $(i+1)$ toppled one grain to its right neighbor. Any such active site will topple with a probability p, which depends on the energy of the particle, and one grain leaves site i and moves to site $(i+1)$,

$$h(i) \to h(i) - 1,$$

$$h(i+1) \to h(i+1) + 1. \tag{2}$$

When no active site remains on the pile, the avalanche is said to be over. Since the relaxation of each grain depends on the variation of its kinetic and potential energy, we propose the following form of the toppling probability, which depends on the energy difference ΔE between the energy of the moving grain and the energy that it would have if it were motionless at the critical slope $z = z_c$, which we take in what follows to be $z_c = 1$:

$$p(\Delta E) = 1 - \exp(-\Delta E), \tag{3}$$

where E is the total energy of the moving grain, which has two contributions.

(1) The kinetic energy E_c which results from the successive shocks:

$$\Delta E_c \approx \alpha(v^2 - 1), \tag{4}$$

where v represents the average velocity of the moving grain and α the related parameter that incorporates the inertial effects. Indeed, if we assume that the grains move without sliding, it is easy to show that the inertiaterm is proportional to v^2. If we consider the position $x(i)$ on the surface of the pile of the toppling grain located at site i at time $t(i)$, we can show that the average velocity may be calculated as follows:

$$v = \frac{\Delta x}{\Delta t} = \frac{\sum_i [x(i) - x(i+1)]}{\sum_i [t(i+1) - t(i)]} \approx \frac{\sum_i [h(i) - h(i+1)]}{n_i}, \quad (5)$$

where n_i is the number of topplings a given grain has done before reaching the site i, and the sum is over the history of the toppling grain. Hence, we introduce a form of memory into the system in which the velocity of toppling grains evolves throughout the avalanche. The falling grains gain momentum, and if the process persists long enough they can be stopped only at the boundary of the system.

(2) The potential energy, which is proportional to the height difference, $\Delta E_p = \gamma[z(i) - 1]$. The parameter γ incorporates the frictional effects implicitly.

In addition to the height $h(i)$ defined at each site, the configuration of the system involves also the velocities of the grains. Hence, for a full specification of the rules governing the motion of grains in the model, we have to define precisely how the unstable grains move under toppling and in which order they topple. The system evolves under a fully parallel update dynamics in which all unstable sites topple one grain at the same time to their neighbors. In order to clarify the parallel update process used in our model and the number of moving grains at each time step, we start from a situation where only the sites $(i+1)$ and $(i-1)$ are active. The transfer of two grains from $(i+1)$ and $(i-1)$ to their neighbors is done at the same time. This constitutes a single microstep of evolution. In this process the grain of each site moves to the right. Thus, the sites $(i+2)$ and $(i-2)$ are perturbed once and the site (i) twice, leading to one possible toppling at the sites $(i+2)$ and $(i-2)$ and two topplings at the site (i) in the next microstep. Consequently, we remark that the toppling process may involve more than one grain per site at each microstep time. On the other hand, the velocity of each transferred grain is updated following Eq. (5).

The model we suggest is controlled by the following dynamics.

(1) One grain is deposited on the top of the pile (site $i = 1$).

(2) If the slope $z(i)$, i.e., the height difference, of the site $i=1$ exceeds a specified threshold value $z=z_c$ one grain, to which we assign a velocity v obtained from Eq. (5), is toppled with a probability $p(\Delta E)$ to its nearest neighbor and the avalanching process starts.

(3) Following a parallel update, all sites are visited at the same time. For each microstep of evolution a velocity, which is calculated from Eq. (5), is assigned to the moving grain located at site i. Its energy is then calculated and it relaxes with a probability $p(\Delta E)$ to site $(i+1)$ if its local slope $z > z_c$. The grain stops if $z \leqslant z_c$.

(4) If the grain of the visited site is motionless and its local slope exceeds z_c, it loses only potential energy since $\Delta E_c = 0$ and it flips with a probability $p(\Delta E)$.

(5) The process is continued until no active site appears and we restart from the first step 1.

The friction effects are introduced implicitly in the toppling probability $p(\Delta E)$, which is a function of the height difference and the kinetic energy accumulated by a grain of rice during a sequence of topples. As in realistic situations we suppose that grains with high accumulated energy and large height have more chance to topple, which is in agreement with the chosen exponential form of the toppling probability.

III. SIMULATION RESULTS AND DISCUSSION

We study the model in the slowly driven limit where the rate of deposition is slow. Any avalanche that might be started by deposition of a grain will have ended before a new grain is added. The simulations of our model show the existence of a critical behavior with continuous variation of critical exponents depending on the physical properties of the grains. Our analysis is based on the study of different quantities related to the avalanche dynamics, namely, the avalanche size s, which is the total number of topplings during the process, the lifetime of the avalanche T, which is the duration of the relaxation process, the size of the discharge events m, which is the number of grains that leave the system from the boundary, and the residence time T_r, which is defined as the time spent in the system by an added grain.

A. Avalanches

We start by investigating the distribution of avalanche sizes s, which follows a scaling form,

$$P(s,L) = s^{-\tau} f(s/L^{\nu}). \quad (6)$$

τ and ν are critical exponents and f is a scaling function rapidly decaying for large arguments. Alternatively, we have $P(s,L) = L^{-\beta} g(s/L^{\nu})$. In order to reduce the finite-size effects on our simulation data and increase the accuracy of the determination of avalanche exponents, we use the function $H(s, L_1, L_2)$ [21] defined as

$$H(s,L_1,L_2) = \frac{P(s,L_1)^{\ln(L_1)}}{P(s,L_2)^{\ln(L_2)}}, \quad (7)$$

where L_1 and L_2 are two different system sizes. The nice property of this function is that, in contrast to the probability distribution, the numerical estimate of the critical exponents related to the avalanche properties can be determined with high accuracy, since the size effects vanish for large sizes, and then any crossover behavior at large values of s will appear as a sharp variation.

The size effects on the distribution $P(s,L)$ [Fig. 1(a)] are studied for different values of the parameters α and γ. Using the scaling analysis, we show that our system presents a nonuniversal behavior such that the equality $\tau = \beta/\nu$ is violated. In Figs. 1(b)–1(d) we show that for $\alpha=0.1$ and $\gamma=2$, $\alpha=0.01$ and $\gamma=0.7$, and $\alpha=0.001$ and $\gamma=0.1$, respectively, the finite-size scaling works extremely well. The resulting critical exponents for $\alpha=0.1$ and $\gamma=2$, for example, are $\beta = 3.15$, $\nu=2.1$, and $\tau=1.74$, which is different from the ratio β/ν. For high values of the parameters α and γ, $p \rightarrow 1$, the system is highly activated and only large-size avalanches are

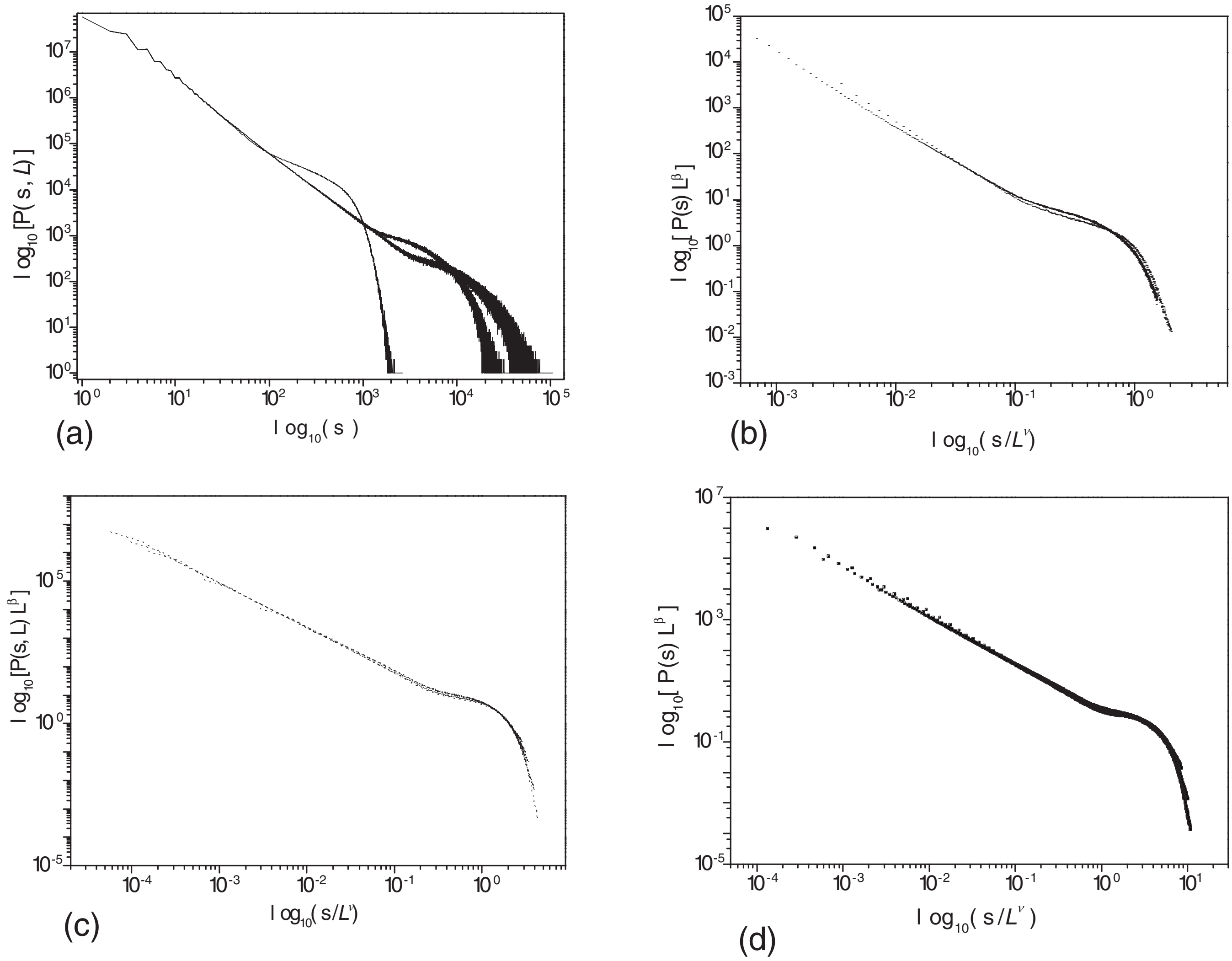

FIG. 1. (a) Log_{10}-$\log_{10}$ plot of the avalanche size probability distribution $P(s,L)$. Simulations were performed for several system sizes for $\alpha=0.1$ and $\gamma=2$. The best finite-size scaling fit is obtained with the scaling exponents (b) $\nu=2.1$ and $\beta=3.15$ for $\alpha=0.1$ and $\gamma=2$, (c) $\nu=2.12$ and $\beta=3.34$ for $\alpha=0.01$ and $\gamma=0.7$, and (d) $\nu=2.15$ and $\beta=3.33$ for $\alpha=0.001$ and $\gamma=0.1$.

frequent. As a result, the SOC behavior is lost and the $\log_{10}$-$\log_{10}$ plot of the avalanche size distribution $P(s,L)$ deviates visibly from power law behavior, which is in agreement with previous results [18–20]. However, we restrict ourselves to the parameter values $0 \leqslant \alpha \leqslant 3$ and $0 < \gamma \leqslant 3$ where the power law behavior is clearly observed. Within this region of the parameter space we find that the maximal value of τ that may be obtained without losing the SOC behavior is $\tau \sim 1.89$. It approaches the experimental critical exponent value, $\tau \sim 2.02$.

The results obtained from our numerical simulations show that our model presents a nonuniversal behavior. The critical exponents exhibit a continuous variation versus the parameters of the system α and γ (Fig. 2). By fixing the value of α and varying γ, the critical exponent τ decreases for low values of γ while for high values it is an increasing function. This result may be understood by the fact that for low values of γ, i.e., high friction, the system may develop relatively large slopes with a large potential energy leading to a large number of metastable state sites. Thus, with an increase in the value of γ the toppling probability p increases, leading to

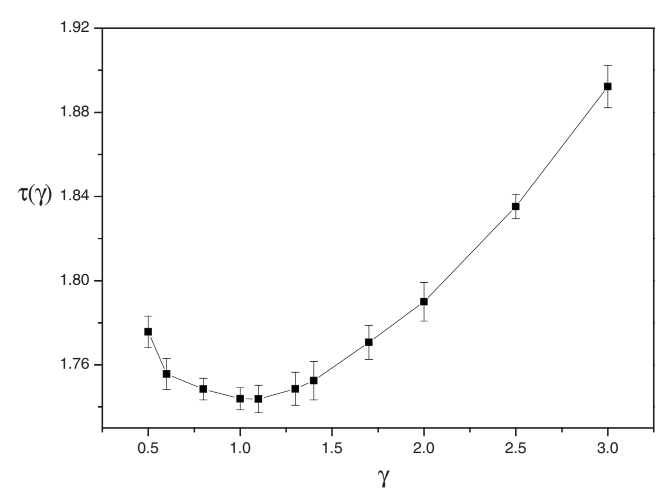

FIG. 2. Variation of the critical exponent τ versus the parameter γ related to the potential energy for $L=200$ and $\alpha=0.1$. The γ-dependent corrections to scaling are indicated by error bars.

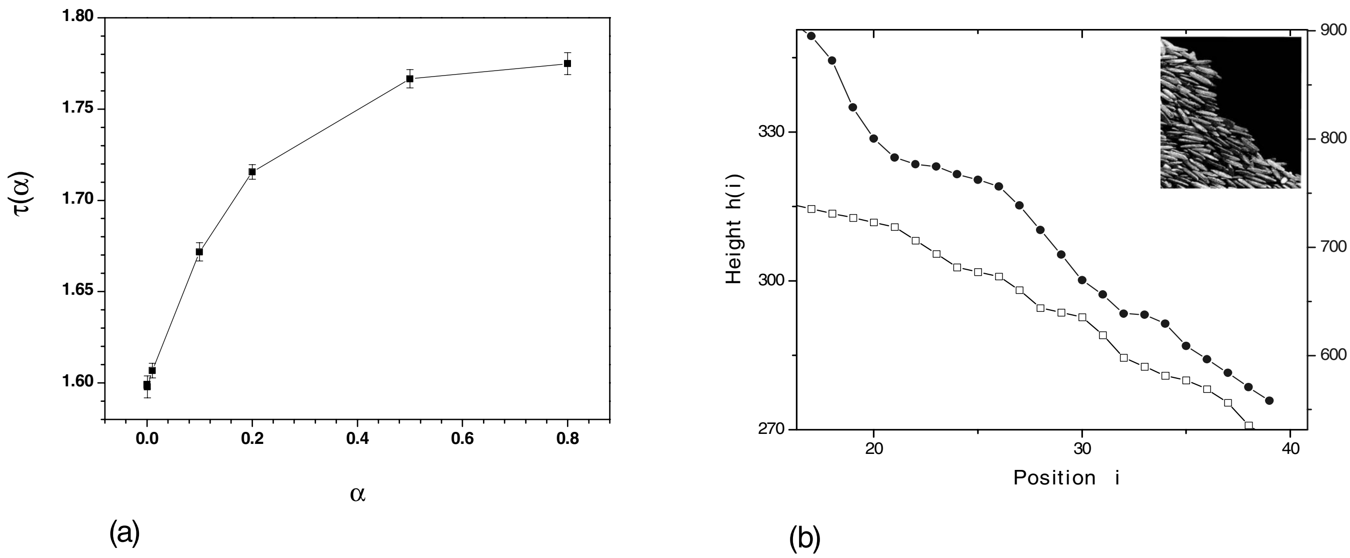

FIG. 3. (a) Critical exponent τ versus the parameter α related to the kinetic energy for system size $L=200$ and $\gamma=1$. The error bars indicate the γ-dependent corrections to scaling. (b) Profile of the pile for system size $L=40$ and two different values of α. The pile develops a large slope, signaling the presence of packing configurations when α increases from $\alpha=0.01$ (open squares) to 0.1 (closed circles). The inset presents a close-up photograph of the rice pile in Ref. [10], where the local slope varies considerably along the pile.

more frequent high numbers of discharges. As a result, big avalanche sizes are favored and the slope of $\log_{10}[P(s,L)]$ increases, i.e., τ decreases. On the other hand, for high values of γ, i.e., low friction, the metastable states occurring in the system are reduced since $p \to 1$. Hence, low numbers of discharges are probable in each avalanche process. Consequently, the more γ increases the more frequent are small avalanches, which leads to an increase of the critical exponent τ.

On the other hand, by fixing the parameter γ we show that τ is an increasing function of α [Fig. 3(a)]. This is due to the fact that, for low inertia where the frictional contact is enhanced, i.e., low values of α, the system develops metastable states responsible for the packing states [Fig. 3(b)] observed experimentally [10]. As a result, the local slope varies considerably along the profile and we observe that the larger is α the rougher is the profile of the pile. These patterns of steepness in the profile change with changed parameters of the system. Consequently, small avalanches are more frequent than large ones, which reinforces the value of the critical exponent τ. We note that for $\alpha \to \infty$ and/or $\gamma \to \infty$ one gets a trivial model: $z_i=1$ for all i and any avalanche reaches the edge of the pile. The SOC is then lost.

It is well established [5,15] that the average number of topplings for a given grain before being discharged scales as

$$\langle s \rangle \sim L^q. \tag{8}$$

Using Monte Carlo simulations we show that for our model the critical exponent $q=1$ independently of the parameter values and the details of the sandpile model [Fig. 4(a)]. This

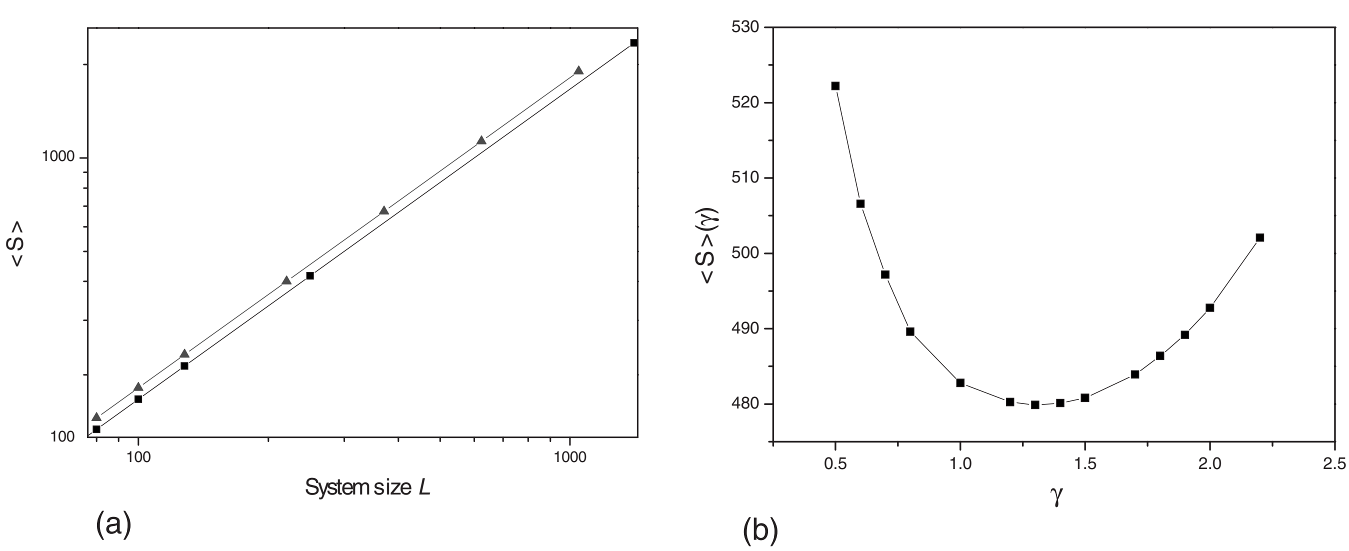

FIG. 4. (a) Mean avalanche size $\langle s \rangle$ versus the system size L for $\alpha=0.1$ and for $\gamma=1$ (squares) and 0.5 (triangles). $\langle s \rangle$ scales as L for different values of the system parameters. (b) Mean avalanche size $\langle s \rangle$ versus γ for $L=200$ and $\alpha=0.1$.

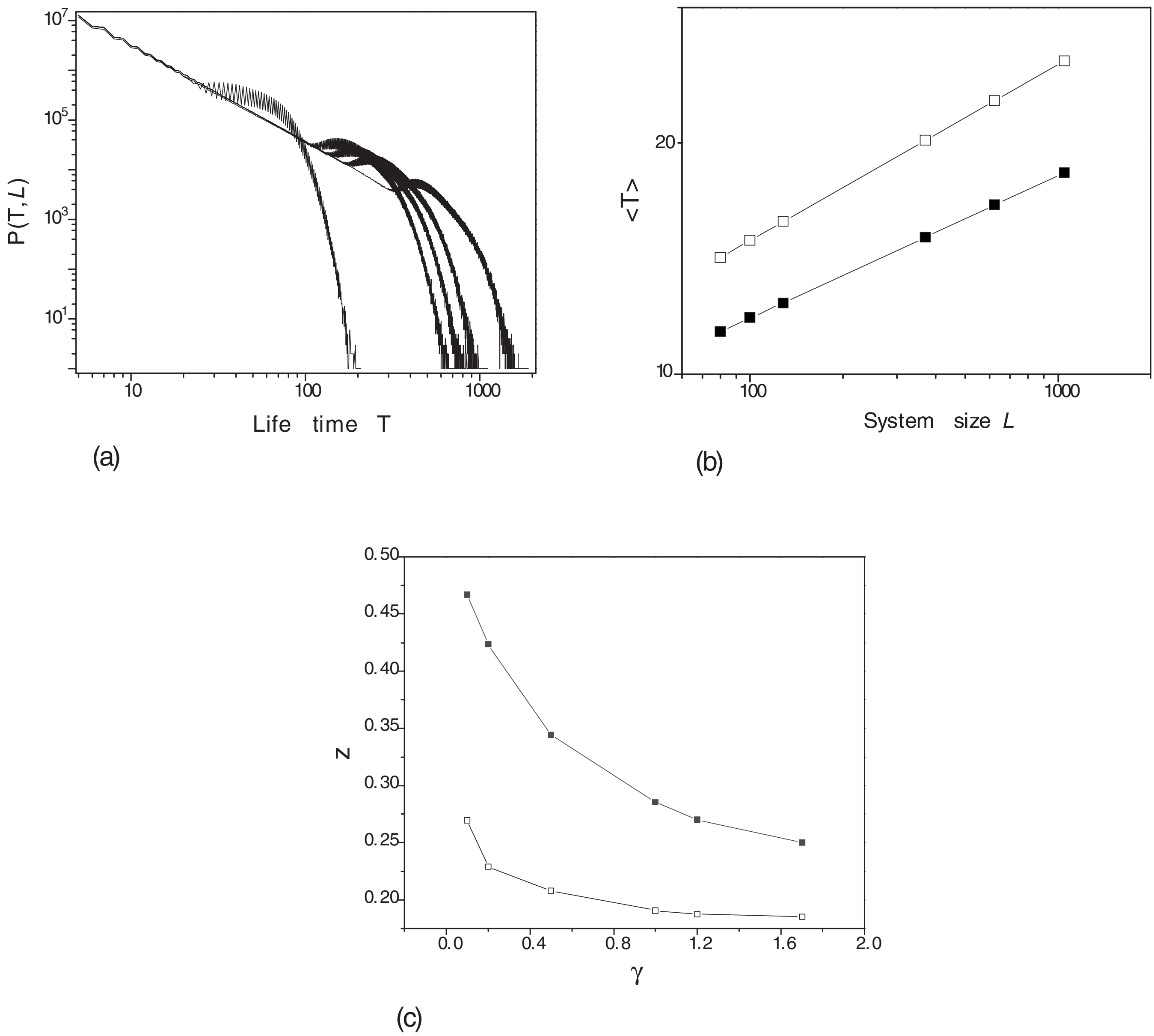

FIG. 5. (a) Log-log plot of the avalanche lifetime probability distribution $P(T,L)$. Simulations were performed for several system sizes for $\alpha=0.1$ and $\gamma=1$. (b) Mean avalanche lifetime $\langle T\rangle$ vs the system size L for $\alpha=0.1$ and for two values of γ. The empty squares represent $\gamma=0.2$ and the full squares $\gamma=1.7$. (c) Variation of the critical exponent z related to the mean avalanche lifetime $\langle T\rangle$ vs γ for $L=200$ and two values of the parameter $\alpha=0.1$ (full squares) and 0.5 (empty squares).

result agrees with previous work [5,15] and emphasizes the fact that q is a universal critical exponent at least for one-dimensional systems. But the variation of the mean avalanche size versus γ [Fig. 4(b)] follows the same variation as the critical exponent τ.

To further test our conclusion regarding the variation of the critical exponents, we have studied different definitions of avalanche size. When using the lifetime T of the avalanche we find that the probability density function is well described by the scaling form

$$P(T,L) = T^{-y} f(T/L^{\sigma}) \tag{9}$$

where y and σ are critical exponents. In Fig. 5(a) we show the distribution of lifetimes for different system sizes. The study of the scaling effect shows that the critical exponent y presents the same variation versus α and γ as was quoted above for the critical exponent τ. On the other hand, the average value of the lifetime [Fig. 5(b)] scales as

$$\langle T\rangle \sim L^{z}, \tag{10}$$

where the value of the exponent z depends on the values of the system parameters [Fig. 5(c)].

B. Residence time

The residence time of grains through a pile of rice has been studied both theoretically and experimentally. The time unit is the duration between successive additions of two sand

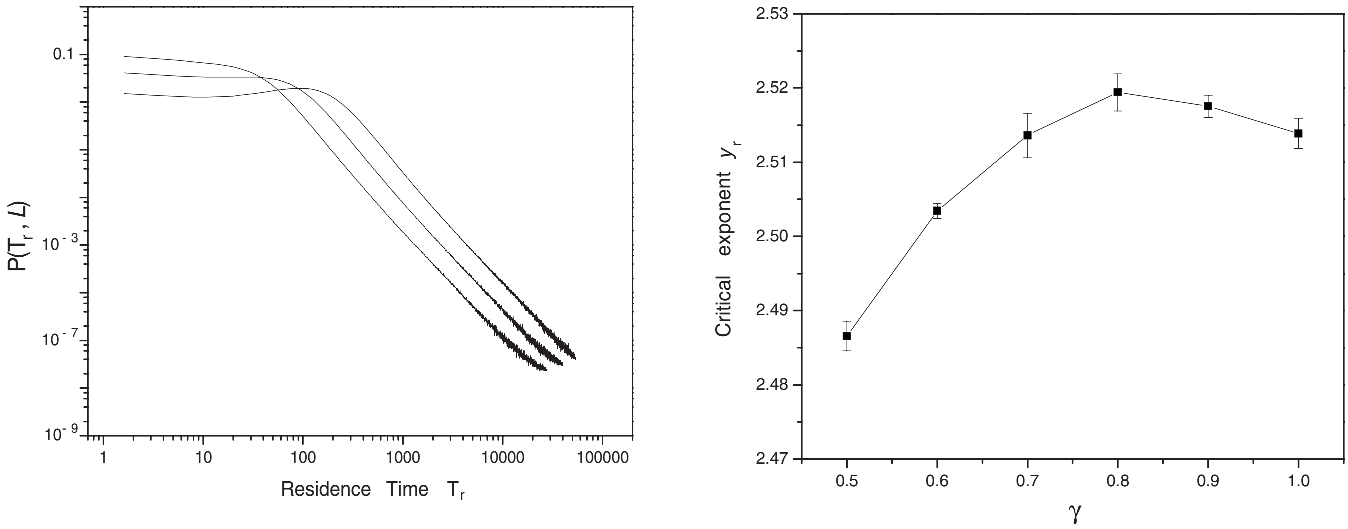

FIG. 6. (a) Log_{10}-$\log_{10}$ plot of the probability distribution of the residence time for $\alpha=0.1$, $\gamma=0.5$, and with system sizes $L=80$, 160, and 320. The data have been averaged over exponentially increasing bins with base 1.4. (b) Variation of the critical exponent y_r versus γ for $\alpha=0.1$ and $L=160$. The γ-dependent corrections to scaling are indicated by error bars.

grains. The residence time of a sand grain is defined as the time spent by that grain inside the system. We calculate the probability distribution $P(T_r,L)$ of the residence time T_r after the pile has reached the stationary critical state and for different values of α and γ [Fig. 6(a)]. It obeys a finite-size scaling form:

$$P(T_r,L) = T_r^{-y_r} g(T_r/L^{\sigma_r}), \tag{11}$$

and is essentially constant for small values of T_r, exhibiting a crossover to a decaying power law behavior for large residence times with varying critical exponent y_r. We note that such behavior was observed experimentally [4]. The critical exponents y_r and τ present opposite variations [Fig. 6(b)]. This is due to dependence of the depth of the active zone on the parameters α and γ. However, when the depth of the active zone increases, the time that the grains spend in the system decreases, which produces an increase of the critical exponent y_r. Furthermore, the decrease of τ means that large avalanches occur with increasing probability, and hence deeply buried grains tend to leave the system for a short time, which leads to an increase of the critical exponent y_r values, and vice versa. It is worthwhile to note that by a combination of combining numerical simulations and scaling arguments, it was shown for the Christensen *et al.* rice-pile model [22] that the distribution probability of residence time $T_j>t$ at site j, $P(T_j>t)$, does not obey a simple finite-size scaling and the corresponding critical exponent y_r may be derived in a simple way from its analytical expression. One finds $y_r=2$.

TABLE I. The variation of the critical exponent χ related to the average residence time ($\langle T_r\rangle \sim L^{\chi}$) versus γ for $\alpha=0.1$.

γ	0.5	0.6	0.7	0.8	0.9	1.5	1.7
χ	1.204	1.223	1.245	1.269	1.324	1.484	1.543

Our numerical investigations show that the average residence time exhibits, for fixed time series, a power law behavior according to the system size, $\langle T_r\rangle \propto L^{\chi}$, where the values of χ depend on the parameters of the system, α and γ (Table I). On the other hand, it was found, using an analytical approach for sandpile models [11], that $\chi=2$, while the experiment gives $\chi=1.5\pm0.2$ and some suggested rice-pile models [4,23] found that $\chi=1.3$. However, we argue numerically that when the time simulation steps t increase, unlike the first moment of the size and the lifetime, the average residence time presents a logarithmic increase versus t (Fig. 7),

$$\langle T_r\rangle \propto \log_{10}(t). \tag{12}$$

This result comes from the fact that the deeply buried grains had to be released by large avalanches in order to

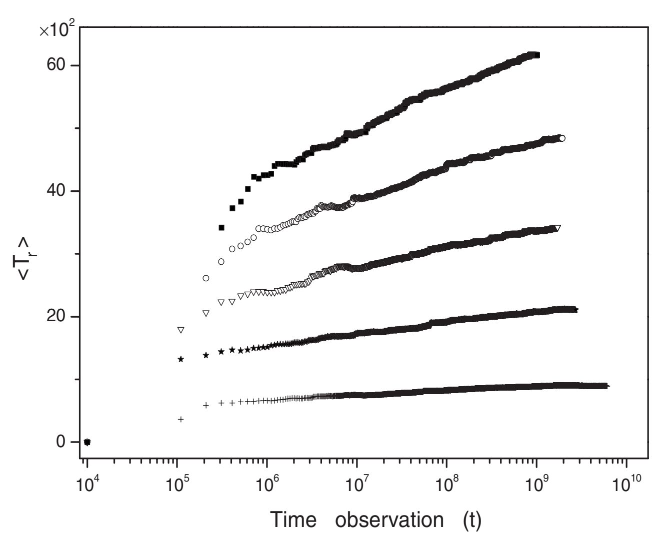

FIG. 7. Semilogarithmic plot of the first moment of the residence time for different system sizes and for $\alpha=0.1$ and $\gamma=0.5$.

move on. However, broad avalanches are not very frequent. Hence, deeply buried grains have a tendency to stay a very long time in the system. Therefore these particles do not contribute to the mean residence time because they come out only for large numbers of simulation time steps (about 10^{15} time steps). Thus, we can prove numerically, as suggested by Dhar [11], that the numerical underestimate of the mean residence time is due to the short time observation, making $\langle T_r \rangle$ unreliable.

The residence time depends on the model and the microscopic details that it incorporates, since the avalanche dynamics is governed by the stochastic toppling parameter p. Thus, we think that the discrepancy in the estimate of the first moment of the residence time probability distribution between our results and the others obtained analytically and numerically is presumably due to the time taken by the grains to leave the system.

IV. CONCLUSION

In conclusion, we have simulated a one-dimensional cellular automaton with local, limited, and stochastic dynamics. The model we have suggested takes into account the kinetic and potential energy of the grains. The inertial effects, which are incorporated implicitly in the kinetic energy term, have been investigated. We have shown that the model describes a nonuniversal behavior where the critical exponents change continuously when the parameters associated with the kinetic and potential energies are varied. The dynamics of the system has an important dependence on the dissipation level and exhibits SOC behavior in a certain range of the adjustable parameters. All features of SOC were observed: finite-size scaling and power law behavior of all quantities that characterize the avalanche. In contrast to previous rice-pile models [5,15,16], we found that by using a stochastic toppling probability p, which depends on the kinetic and potential energy of the toppling grain, the distribution size critical exponent is improved, $\tau = 1.89$, and it approaches the experimental one, $\tau \sim 2.02$. On the other hand, the distribution of residence time and its related critical exponent agree both qualitatively and quantitatively with the experimental findings, where $y_r = 2.5 \pm 0.2$. However, the mean residence time does not depend either on any details of the toppling rules or on microscopic details of the model, and the proof presented in Ref. [11] that it scales as L^2 remains valid for our model also. Furthermore, the study of the friction effects [24], in an explicit way, on a rice-pile model shows that SOC occurs in a slowly driven granular pile of rice as long as friction dominates inertial effects. But for high values of the adjustable parameters the SOC is lost.

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